

# CHENG-YAU OPERATOR AND GAUSS MAP OF SURFACES OF REVOLUTION

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**ABSTRACT.** We study the Gauss map  $G$  of surfaces of revolution in the 3-dimensional Euclidean space  $\mathbb{E}^3$  with respect to the so called Cheng-Yau operator  $\square$  acting on the functions defined on the surfaces. As a result, we establish the classification theorem that the only surfaces of revolution with Gauss map  $G$  satisfying  $\square G = AG$  for some  $3 \times 3$  matrix  $A$  are the planes, right circular cones, circular cylinders and spheres.

## 1. INTRODUCTION

The theory of Gauss map of a surface in a Euclidean space and a pseudo-Euclidean space is always one of interesting topics and it has been investigated from the various viewpoints by many differential geometers ([2, 3, 8, 9, 10, 6, 11, 13, 14, 15, 16, 18, 19, 20, 21]).

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Let  $M$  be a surface of the Euclidean 3-space  $\mathbb{E}^3$ . The map  $G : M \rightarrow S^2 \subset \mathbb{E}^3$  which sends each point of  $M$  to the unit normal vector to  $M$  at the point is called the *Gauss map* of the surface  $M$ , where  $S^2$  is the unit sphere in  $\mathbb{E}^3$  centered at the origin. It is well known that  $M$  has constant mean curvature if and only if  $\Delta G = \|dG\|^2 G$ , where  $\Delta$  is the Laplace operator on  $M$  corresponding to the induced metric on  $M$  from  $\mathbb{E}^3$  ([23]). Surfaces whose Gauss map is an eigenfunction of Laplacian, that is,  $\Delta G = \lambda G$  for some constant  $\lambda \in \mathbb{R}$ , are the planes, circular cylinders and spheres ([6]).

Generalizing this equation, F. Dillen, J. Pas and L. Verstraelen ([11]) studied surfaces of revolution in a Euclidean 3-space  $\mathbb{E}^3$  such that its Gauss map  $G$  satisfies the condition

$$(1.1) \quad \Delta G = AG, \quad A \in R^{3 \times 3}.$$

As a result, they proved ([11])

**Proposition 1.1.** Among the surfaces of revolution in  $\mathbb{E}^3$ , the only ones whose Gauss map satisfies (1.1) are the planes, the spheres and the circular cylinders.

Baikoussis and Blair also studied ruled surfaces in  $\mathbb{E}^3$  and proved ([2])

**Proposition 1.2.** Among the ruled surfaces in  $\mathbb{E}^3$ , the only ones whose Gauss map satisfies (1.1) are the planes and the circular cylinders.

Generalized slant cylindrical surfaces (GSCS's) are natural extended notion of surfaces of revolution ([17]). Surfaces of revolution, cylindrical surfaces and tubes along a plane curve are special cases of GSCS's. In [19], the first author and B. Song proved that among the GSCS's in  $\mathbb{E}^3$ , the only ones whose Gauss map satisfies (1.1) are the planes, the spheres and the circular cylinders.

The so-called Cheng-Yau operator  $\square$  (or,  $L_1$ ) is a natural extension of the Laplace operator  $\Delta$  (cf. [1], [7]). Hence, following the condition (1.1), it is natural to ask as follows.

**Question 1.3.** Among the surfaces of revolution in a Euclidean 3-space  $\mathbb{E}^3$ , which one satisfy the following condition?

$$(1.2) \quad \square G = AG, \quad A \in R^{3 \times 3}.$$

In this paper, we give a complete answer to the above question.

Throughout this paper, we assume that all objects are smooth and connected, unless otherwise mentioned.

## 2. CHENG-YAU OPERATOR AND LEMMAS

Let  $M$  be an oriented surface in  $E^3$  with Gauss map  $G$ . We denote by  $S$  the shape operator of  $M$  with respect to the Gauss map  $G$ . For each  $k = 0, 1$ , we put  $P_0 = I, P_1 = \text{tr}(S)I - S$ , where  $I$  is the identity operator acting on the tangent bundle of  $M$ . Let us define an operator  $L_k : C^\infty(M) \rightarrow C^\infty(M)$  by  $L_k(f) = -\text{tr}(P_k \circ \nabla^2 f)$ , where  $\nabla^2 f : \chi(M) \rightarrow \chi(M)$  denotes the self-adjoint linear operator metrically equivalent to the hessian of  $f$ . Then, up to signature,  $L_k$  is the linearized operator of the first variation of the  $(k+1)$ -th mean curvature arising from normal variations of the surface. Note that the operator  $L_0$  is nothing but the Laplace operator acting on  $M$ , i.e.,  $L_0 = \Delta$  and  $L_1 = \square$  is called the Cheng-Yau operator introduced in [7].

Now, we state a useful lemma as follows ([1]).

**Lemma 2.1.** Let  $M$  be an oriented surface in  $E^3$  with Gaussian curvature  $K$  and mean curvature  $H$ . Then, the Gauss map  $G$  of  $M$  satisfies

$$(2.1) \quad \square G = \nabla K + 2HKG,$$

where  $\nabla K$  denotes the gradient of  $K$ .

Now, using Lemma 2.1 we give some examples of surfaces with Gauss map satisfying (1.2).

**Examples.**

(1) Flat surfaces. In this case, we have  $\square G = 0$ , and hence flat surfaces satisfy  $\square G = AG$  for some  $3 \times 3$  matrix  $A$ . Note that the matrix  $A$  must be singular.

(2) Spheres:  $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$ . In this case, we have  $G = \frac{1}{r}(x - a, y - b, z - c)$  so the sphere satisfies  $\square G = AG$  with  $A = \frac{-1}{r^3}I$ , where  $I$  denotes the identity matrix.

### 3. GAUSS MAP OF SURFACES OF REVOLUTION

We consider a unit speed plane curve  $C : (x(s), 0, z(s))$  with  $x(s) > 0$  in the  $xz$  plane which is defined on an interval  $I$ . By rotating the curve  $C$  around  $z$ -axis, we get a surface of revolution  $M$ , which is parametrized by

$$(3.1) \quad X(s, t) = (x(s) \cos t, x(s) \sin t, z(s)).$$

The adapted frame field  $\{e_1, e_2, G\}$  on the surface of revolution  $M$  are given by

$$\begin{aligned}
 e_1 &= X_s = (x'(s) \cos t, x'(s) \sin t, z'(s)), \\
 (3.2) \quad e_2 &= \frac{1}{x} X_t = (-\sin t, \cos t, 0), \\
 G &= e_1 \times e_2 = (-z' \cos t, -z' \sin t, x').
 \end{aligned}$$

The principal curvatures  $k_1, k_2$  of  $M$  with respect to the Gauss map  $G$  are respectively ([12])

$$\begin{aligned}
 (3.3) \quad k_1 &= \langle S(e_1), e_1 \rangle = x'z'' - x''z' = \kappa, \\
 k_2 &= \langle S(e_2), e_2 \rangle = \frac{z'}{x},
 \end{aligned}$$

where  $S$  and  $\kappa$  denote the shape operator of  $M$  and the plane curvature of the plane curve  $C$ , respectively.

Since the parametrization  $(x(s), 0, z(s))$  of the plane curve  $C$  is of unit speed, there exists a smooth function  $\theta = \theta(s)$  such that  $x' = \cos \theta$  and  $z' = \sin \theta$ . Then, the Gaussian curvature  $K$  and the mean curvature  $H$  of  $M$  are, respectively, given by

$$\begin{aligned}
 (3.4) \quad K &= k_1 k_2 = \frac{\theta'(s) \sin \theta}{x}, \\
 2H &= k_1 + k_2 = \theta'(s) + \frac{\sin \theta}{x}.
 \end{aligned}$$

Hence, the gradient  $\nabla K$  of the Gaussian curvature  $K$  of  $M$  is given by

$$(3.5) \quad \nabla K = K'(s)e_1,$$

where

$$(3.6) \quad K'(s) = \frac{1}{x^2} \{x\theta''(s) \sin \theta + x\theta'(s)^2 \cos \theta - \theta'(s) \cos \theta \sin \theta\}.$$

We now suppose that the Gauss map  $G$  of the surface of revolution  $M$  satisfies for a  $3 \times 3$  matrix  $A = (a_{ij})$

$$(3.7) \quad \square G = AG.$$

Recall that the Gauss map  $G$  is given by

$$(3.8) \quad G(s, t) = (-\sin \theta \cos t, -\sin \theta \sin t, \cos \theta).$$

Then, it follows from (2.1), (3.4) and (3.5) that

$$(3.9) \quad \begin{aligned} & \{K'(s) \cos \theta - 2KH \sin \theta\} \cos t \\ &= -a_{11} \sin \theta \cos t - a_{12} \sin \theta \sin t + a_{13} \cos \theta, \end{aligned}$$

$$(3.10) \quad \begin{aligned} & \{K'(s) \cos \theta - 2KH \sin \theta\} \sin t \\ &= -a_{21} \sin \theta \cos t - a_{22} \sin \theta \sin t + a_{23} \cos \theta \end{aligned}$$

and

$$(3.11) \quad K'(s) \sin \theta + 2KH \cos \theta = -a_{31} \sin \theta \cos t - a_{32} \sin \theta \sin t + a_{33} \cos \theta.$$

First, we suppose that the set  $J = \{s \in I \mid \theta'(s) \neq 0\}$  is nonempty. Then  $\theta(I)$  contains an interval, hence we get from (3.9)-(3.11) that  $a_{12} = a_{13} = a_{21} = a_{23} = a_{31} = a_{32} = 0$  and  $a_{11} = a_{22}$ . Thus we obtain  $A = \text{diag}(\lambda, \lambda, \mu)$ ,

$$(3.12) \quad K'(s) \cos \theta - 2KH \sin \theta = -\lambda \sin \theta,$$

and

$$(3.13) \quad K'(s) \sin \theta + 2KH \cos \theta = \mu \cos \theta.$$

Note that (3.12) and (3.13) are equivalent to the following:

$$(3.14) \quad K'(s) = a \cos \theta \sin \theta,$$

and

$$(3.15) \quad 2KH = -a \sin^2 \theta + \mu,$$

where we put  $a = \mu - \lambda$ .

We prove the following lemma, which plays a crucial role in the proof of our main theorem.

**Lemma 3.1.** Let  $M$  be a surface of revolution given by (3.1) with nonempty set  $J = \{s \in I \mid \theta'(s) \neq 0\}$ . Suppose that the Gauss map  $G$  of  $M$  satisfies  $\square G = AG$  for some  $3 \times 3$  matrix  $A$ . Then  $A$  is of the form  $\lambda I$ , where  $I$  is an identity matrix.

**Proof.** The above discussions show that  $A$  is a diagonal matrix of the form  $A = \text{diag}(\lambda, \lambda, \mu)$  for some constants  $\lambda$  and  $\mu$ . We put  $a = \mu - \lambda$ . Then, it follows from (3.4), (3.6), (3.14) and (3.15) that

$$(3.16) \quad x\theta''(s) \sin \theta + x\theta'(s)^2 \cos \theta - \theta'(s) \cos \theta \sin \theta = ax^2 \cos \theta \sin \theta$$

and

$$(3.17) \quad x\theta'(s)^2 \sin \theta + \theta'(s) \sin^2 \theta = (-a \sin^2 \theta + \mu)x^2.$$

By differentiating the both sides of (3.17) with respect to  $s$ , we get

$$(3.18) \quad \begin{aligned} & \theta''(s) \sin^2 \theta + 2x\theta'(s)\theta''(s) \sin \theta + x\theta'(s)^3 \cos \theta + 3\theta'(s)^2 \sin \theta \cos \theta \\ & + 2ax^2\theta'(s) \sin \theta \cos \theta = 2x \cos \theta (-a \sin^2 \theta + \mu). \end{aligned}$$

If we substitute  $\theta''(s)$  in (3.16) into (3.18), then we have

$$(3.19) \quad \begin{aligned} & -x^2\theta'(s)^3 \cos \theta + 4x\theta'(s)^2 \cos \theta \sin \theta + \{\cos \theta \sin^2 \theta + 4ax^3 \cos \theta \sin \theta\}\theta'(s) \\ & + 3ax^2 \cos \theta \sin^2 \theta - 2\mu x^2 \cos \theta = 0. \end{aligned}$$

Let us substitute  $\theta'(s)^2$  in (3.17) into (3.19). Then we obtain

$$(3.20) \quad \begin{aligned} & 5x\theta'(s)^2 \cos \theta \sin \theta + \{\cos \theta \sin^2 \theta + 5ax^3 \cos \theta \sin \theta - \mu x^3 \cot \theta\} \theta'(s) \\ & + 3ax^2 \cos \theta \sin^2 \theta - 2\mu x^2 \cos \theta = 0. \end{aligned}$$

Once more, we substitute  $\theta'(s)^2$  in (3.17) into (3.20). Then we get

$$(3.21) \quad \theta'(s) = \frac{\gamma x^2}{\alpha x^3 + \beta},$$

where we put

$$(3.22) \quad \begin{aligned} \alpha(s) &= -5a \sin^2 \theta(s) + \mu, \beta(s) = 4 \sin^3 \theta(s), \\ \gamma(s) &= -2a \sin^3 \theta(s) + 3\mu \sin \theta(s). \end{aligned}$$

Now, we replace  $\theta'(s)$  in (3.17) with that in (3.21). Then we have

$$(3.23) \quad a_6 x^6 + a_3 x^3 + a_0 = 0,$$

where we use the following notations:

$$(3.24) \quad \begin{aligned} a_6(\theta) &= 25a^3 \cos^6 \theta + 5a^2(15\lambda - 8\mu) \cos^4 \theta \\ &+ a(5a - \mu)(4\mu - 15\lambda) \cos^2 \theta + \lambda(5a - \mu)^2, \end{aligned}$$

$$(3.25) \quad a_3(\theta) = 26a^2 \sin^7 \theta - 19a\mu \sin^5 \theta - 4\mu^2 \sin^3 \theta$$

and

$$(3.26) \quad a_0(\theta) = -8a \sin^8 \theta + 4\mu \sin^6 \theta.$$

Let us differentiate (3.23) with respect to  $s$ . Here, we denote by  $\dot{a}_i(\theta)$  the derivative of  $a_i(\theta)$  with respect to  $\theta$ ,  $i = 0, 3, 6$ . Using  $x' = \cos \theta$  and  $\theta'(s)$  given by (3.21), we get

$$(3.27) \quad b_6 x^6 + b_3 x^3 + b_0 = 0,$$



where we denote

$$(3.28) \quad b_6(\theta) = 6\alpha \cos \theta a_6(\theta) + \gamma \dot{a}_6(\theta),$$

$$(3.29) \quad b_3(\theta) = 3\alpha \cos \theta a_3(\theta) + 6\beta \cos \theta a_6(\theta) + \gamma \dot{a}_3(\theta)$$

and

$$(3.30) \quad b_0(\theta) = 3\beta \cos \theta a_3(\theta) + \gamma \dot{a}_0(\theta).$$

If we compute  $b_i(\theta)$  for  $i = 0, 3, 6$ , then we have

$$(3.31) \quad b_6(\theta) = \{1050a^4 \sin^8 \theta + \sum_{i=0}^6 p_i(\lambda, \mu) \sin^i \theta\} \cos \theta,$$

$$(3.32) \quad b_3(\theta) = \{-214a^3 \sin^9 \theta + \sum_{i=0}^7 q_i(\lambda, \mu) \sin^i \theta\} \cos \theta$$

and

$$(3.33) \quad b_0(\theta) = \{440a^2 \sin^{10} \theta + \sum_{i=0}^8 r_i(\lambda, \mu) \sin^i \theta\} \cos \theta,$$

where  $p_i(\lambda, \mu)$ ,  $q_i(\lambda, \mu)$  and  $r_i(\lambda, \mu)$  are respectively some polynomials in  $\lambda$  and  $\mu$ .

Eliminating  $x^6$ , it follows from (3.23) and (3.27) that

$$(3.34) \quad c_3 x^3 + c_0 = 0,$$

where

$$(3.35) \quad c_3 = a_3 b_6 - b_3 a_6, c_0 = a_0 b_6 - b_0 a_6.$$

Due to (3.24)-(3.26) and (3.31)-(3.33), we may compute  $c_3$  and  $c_0$  as follows:

$$(3.36) \quad c_3 = \{32650a^6 \sin^{15} \theta + \sum_{j=0}^{13} p_{3j}(\lambda, \mu) \sin^j \theta\} \cos \theta,$$

and

$$(3.37) \quad c_0 = \{-19400a^5 \sin^{16} \theta + \sum_{j=0}^{14} p_{0j}(\lambda, \mu) \sin^j \theta\} \cos \theta,$$

where each  $p_{ij}(\lambda, \mu)$  ( $i = 0, 3$ ) is a polynomial in  $\lambda$  and  $\mu$ .

Let us replace  $x^3$  in (3.23) with  $x^3 = -c_0/c_3$  given in (3.34). Then we have

$$(3.38) \quad a_6 c_0^2 - a_3 c_0 c_3 + a_0 c_3^2 = 0.$$

Using (3.31)-(3.33), (3.36) and (3.37), we may compute the leading terms of those in (3.38) as follows:

$$(3.39) \quad \begin{aligned} a_6 c_0^2 &= -25(19400)^2 a^{13} \sin^{40} \theta + \text{lower degree terms in } \sin \theta, \\ a_3 c_0 c_3 &= 26(19400)(32650) a^{13} \sin^{40} \theta + \text{lower degree terms in } \sin \theta, \\ a_0 c_3^2 &= 8(32650)^2 a^{13} \sin^{40} \theta + \text{lower degree terms in } \sin \theta. \end{aligned}$$

Hence we obtain

$$(3.40) \quad \begin{aligned} a_6 c_0^2 - a_3 c_0 c_3 + a_0 c_3^2 &= -17349480000 a^{13} \sin^{40} \theta \\ &+ \text{lower degree terms in } \sin \theta. \end{aligned}$$

Since  $\theta(I)$  contains an interval, together with (3.38), (3.40) shows that  $a$  must be zero. Thus we have  $\mu = \lambda$  and hence  $A = \lambda I$ . This completes the proof.  $\square$

#### 4. MAIN THEOREMS AND COROLLARIES

Finally, we prove the main theorem as follows.

**Theorem 4.1.** Let  $M$  be a surface of revolution. Then the Gauss map  $G$  of  $M$  satisfies  $\square G = AG$  for some  $3 \times 3$  matrix  $A$  if and only if  $M$  is an open part of the following surfaces:

1) a plane,

- 2) a right circular cone,
- 3) a circular cylinder,
- 4) a sphere.

**Proof.** We consider a surface of revolution  $M$  obtained by rotating the unit speed plane curve  $C : (x(s), 0, z(s))$  with  $x(s) > 0$  around  $z$ -axis which is defined on an interval  $I$ .

Suppose that the Gauss map  $G$  of  $M$  satisfies  $\square G = AG$  for some  $3 \times 3$  matrix  $A$ . For a function  $\theta = \theta(s)$  satisfying  $(x'(s), z'(s)) = (\cos \theta(s), \sin \theta(s))$ , let us put  $J = \{s \in I \mid \theta'(s) \neq 0\}$ .

We divide by two cases.

**Case 1.** Suppose that  $J$  is nonempty. Then, as in the proof of Lemma 3.1 we have  $A = \text{diag}(\lambda, \lambda, \mu)$  with  $a = \lambda - \mu$ . Furthermore, Lemma 3.1 shows that  $a = 0$  that is,  $\lambda = \mu$ . Hence it follows from (3.14) and (3.15) that the Gaussian curvature  $K$  is constant and the mean curvature  $H$  satisfies  $2KH = \lambda$ .

If  $\lambda \neq 0$ , then both of  $K$  and  $H$  are nonzero constant. Hence it follows from a well-known theorem (cf. [22]) that  $M$  is an open part of a sphere. Using (3.17) and (3.21) with  $a = 0$ , it can be directly shown that  $\theta'(s)$  is constant and  $x(s) = r \sin \theta$  for a positive constant  $r$ . This shows that the profile curve  $C$  is an open part of a half circle centered on the rotation axis of  $M$ . Thus,  $M$  is an open portion of a round sphere.

If  $\lambda = 0$ , then  $K$  is constant with  $2KH = 0$ . Suppose that  $K \neq 0$ . Then we have  $H = 0$ . But catenoids are the only minimal nonflat surfaces of revolution, of which Gaussian curvature  $K$  are nonconstant. This contradiction shows that  $K = 0$ . Thus  $M$  is a flat surfaces of revolution. Therefore,  $M$  is an open part of a plane, a right circular cone or a circular cylinder.

**Case 2.** Suppose that  $J$  is empty. Then the profile curve  $C$  of  $M$  is a straight line. Thus,  $M$  is an open part of a plane, a right circular cone or a circular cylinder.

The converse is obvious from (2.1).  $\square$

Combining the results of [11, 19], the following characterization theorems can be obtained.

**Corollary 4.2.** Let  $M$  be a surface of revolution. Then the following are equivalent.

- 1)  $M$  is an open part of a round sphere.
- 2) The Gauss map  $G$  of  $M$  satisfies  $\square G = AG$  for some nonsingular  $3 \times 3$  matrix  $A$ .

**Corollary 4.3.** Let  $M$  be a surface of revolution. Then the following are equivalent.

- 1)  $M$  is an open part of a right circular cone.
- 2) The Gauss map  $G$  of  $M$  satisfies  $\square G = AG$  for some  $3 \times 3$  matrix  $A$ , but not satisfies  $\Delta G = AG$  for any  $3 \times 3$  matrix  $A$ .

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